

Sanskrit-Prakrit Interaction in Elementary Mathematics as Reflected in Arabic and Italian Formulations of the Rule of Three – and Something more on the Rule Elsewhere*

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ABSTRACT

Sanskrit sources from Āryabhaṭa to Bhāskara II have a standard formulation of the rule of three. However, it is clear that mathematics must also have been spoken of and performed during this period (and before) in vernacular environments, and that the two levels must have interacted – not least because the erudite astronomer-mathematicians use commercial arithmetic as the introduction to mathematics. But we have no surviving vernacular texts.

From Brahmagupta onward, however, the standard Sanskrit formulation is supplemented by the observation that two of the known magnitudes are similar in kind, and the third dissimilar. This could be an innovation made within the Sanskrit tradition, but comparison with Arabic and Italian medieval sources seems to rule this out. Instead, it must have been current in the commercial community spanning the Indian Ocean and the Mediterranean – but since the Sanskrit scholars are not likely to have borrowed from Arabic traders, also in vernacular commercial arithmetic as practised within India.

So far, the story seems simple and coherent. However, if Latin twelfth-thirteenth-century writings and sources from the late medieval Ibero-Provençal area are taken into account, loose ends turn up that show the simple story not to be the whole story.

Keywords: Rule of 3, Sanskrit mathematics, Indian vernacular mathematics, Arabic mathematics, Abacus mathematics, Ibero-Provençal practical arithmetic, Latin mathematical translations from the Arabic.

To the friends

Mahdi Abdeljaouad and Ulrich Rebstock

An introductory observation

Let me start with a necessary observation on terminology: *the “rule of three” is a rule, not a problem type*. It is a rule for solving linear problems of the type if A corresponds to X , to what will B correspond?

The rule states in one way or the other (but with *this order* of the arithmetical operations) that the answer is $Y = (B \times X)/A$. Analysis of this “one way or the other” will be my main tool in what follows.

There has been a tendency among historians of mathematics to conflate *the rule* and the *problem type*, which has allowed them to find “the rule of three” in ancient Mesopotamia, ancient Egypt, and in the arithmetical epigrams of the *Greek Anthology*. The consequences of this can at best be understood through the folk tale motif of painting white crosses on all the doors of the town once the door of a suspect has been marked in that way: it ensures that the investigator will find nothing. So, I shall stick to etymology and reserve the name “rule” for the rule.

In consequence of this choice we do not find the rule of three in ancient Mesopotamian, Egyptian or Greek mathematics (or in any of those traditions that were directly derived from them before the Middle Ages). It belongs to ancient and medieval India and China, and (derived from India, as we shall see) to the medieval Arabic and Mediterranean world; from the Renaissance onward it also rises to fame in central and western Europe.

India

From Āryabhaṭa in the late fifth century CE onward, Sanskrit mathematicians use a standard terminology for the four magnitudes A , B , X , and Y – see, for instance, [Elfering 1975: 140] (Āryabhaṭa), [Raṅgācārya 1912: 86] (Mahāvīra) and [Colebrooke 1817: 33, 283] (Bhāskara II, Brahmagupta):¹

A : <i>pramāṇa</i> (“measure”)	X : <i>phala</i> (“fruit”)
B : <i>icchā</i> (“wish”)	Y : <i>icchaphāla</i> (“fruit of wish”)

¹The Bakhshālī manuscript makes copious use of the rule (in particular for verifications) and refers to it by the usual name *trairāśika*; but the only time a partial terminology turns up (X 25, ed. [Hayashi 1995: 358, cf. 439]) it is clearly different.

There is, however, a much earlier Sanskrit appearance of the rule, albeit not making use of this terminology and therefore regarded by Sreeramula Rajeswara Sarma [2002: 135] as only “a rudimentary form of the Rule of Three”. It is found in both recensions of the *Vedāṅgajyotiśa* and may thus go back to c. 400 BCE [Pingree 1978: 536]. It states that the “known result is to be multiplied by the quantity for which the result is wanted, and divided by the quantity for which the known result is given”, which as far as its arithmetic is concerned is not rudimentary at all. As pointed out by Sarma, the descriptive terms used – *jñānatarāśi*, “the quantity that is known”, and *jñeya-rāśi*, “the quantity that is to be known” – also turn up in certain later texts.

All of these sources are written in Sanskrit – with the partial exception of the Bakhshālī manuscript, whose language “though intended to be Sanskrit, has been affected to a considerable degree by a dialect or dialects not only on the phonetic level but also on the morphologic level” [Hayashi 1995: 53], and which also (as mentioned) has a deviating terminology for the rule (and uses a standardized linear organization of the terms, which the Sanskrit sources hint at but do not always draw). Āryabhaṭa as well as Brahmagupta present the rule within the context of astronomical treatises, and Bhāskara I and II were mainly astronomers. The very fact that the mathematics they introduce while having an astronomical purpose in mind is largely commercial or otherwise economical shows clearly, however, that it is borrowed from social groups that were distinct from that of the learned Brahmins and thus speakers of some Prakrit or other vernacular.² Mahāvīra, as a Jaina, was already part of an environment engaged in economical life [Thapar 1966: 65], and that exactly he would write a mathematical treatise not asked for by astronomy (though in the solemn language) fits the picture.³ So does, finally, Bhāskara I’s reference to “worldly practise (*lokavyavahāra*)” in connection with his discussion of the rule of three and elsewhere [Keller 2006: I, 107, cf. 12].⁴ Here, as generally, to quote [Sarma 2010: 202], “Sanskrit has [...] absorbed much from the local traditions. Anthropologists recognize today that the so-called ‘Little Traditions’ played a significant role in shaping the ‘Great Tradition’”.

²Since distinctions and precision in this domain are already difficult for the specialist – cf. [Pollock 2006] – I, as a definite non-specialist, shall abstain from proposing any.

³That Mahāvīra was part of a distinct tradition is highlighted by the presence of several layers of Near Eastern/Mediterranean influence in his geometrical chapter – cf. [Høyrup 2004].

⁴Cf. also the reference to “worldly computations (*laukikagaṇita*)” when a problem about walking men is used to illustrate astronomical conjunction computation [Keller 2006: I, 127].

One thing is to deduct that vernacular mathematics must have existed. Another thing is to conclude anything about how it looked. In India as in other places where survival of (mathematical and other) texts relied on repeated copying, non-prestigious written culture had no better survival possibility than oral culture – that is, we depend almost exclusively on indirect evidence in the shape of references and quotations in the prestigious texts.⁵

Returning to the standard Sanskrit presentations of the rule of three, one feature may be a possible reference to vernacular ways (just *barely* possible when seen in the Indian context in isolation – but as we shall see, opening of the geographical horizon changes things). According to Brahmagupta [trans. Colebrooke 1817: 283],

In the rule of three, argument, fruit and acquisition: the first and last terms must be similar.

Bhāskara I gives a somewhat related explanation in his commentary to the *Āryabhaṭīya*: not, however, when commenting upon Āryabhaṭa’s text but only in connection with the first example, with reference to the linear arrangement of the three known terms [ed., trans. Keller 2006: I, 109f],

this has been stated,

“In order to bring about a Rule of Three the wise should know that in the dispositions the two similar (*sadrśa*) (quantities) are at the beginning and the end. The dissimilar quantity (*asadrśa*) is in the middle”.

The initial “has been stated” suggests that Bhāskara quotes an earlier commentary, cf. [Sarma 2002: 137], while the phrase “the wise should know” seems to imply that

⁵One might hope that the strong reliance on memorization in Indian culture would improve the situation for the permanence of oral culture, but even memorization will probably have been reserved for prestigious cultural items – or at least have been selective, as illustrated by John Warren’s observation in c. 1825 of a Tamil calendar maker who computed “a lunar eclipse by means of shells, placed on the ground, and from tables memorized ‘by means of certain artificial words and syllables’” [Neugebauer 1952: 253]. It is next to certain that ethnomathematical field work will still be able to find surviving sub-scientific mathematical traditions (including their riddles), but the extent to which these are faithful in details to their first-millennium ancestors will be impossible to decide unless they can be connected to parallel sources, such as the “fragments of tables of multiplication, of squares and square roots, and of cubes and cube roots [which] are in Prakrit and must have been in use in the Andhra region at some time” in a Telugu commentary [Sarma 2010: 209]; Sarma relates in parallel that “in Uttar Pradesh, elderly people tell me that they had memorized several multiplication tables of whole numbers and fractions in Vrajbhasa or in Avadhi” (still languages which belong to the second millennium – and perhaps in a form that belongs to the latest century). Everything, we observe, concerns memorized *tables*.

this will be *new* knowledge for the “the wise” (the scholars?): elsewhere in the text, when “the wise” appear, they *know* (pp. 71, 109).⁶

Mahāvīra [trans. Raṅgacārya 1912: 86] explains that

in the rule of three, *Phala* multiplied by *Ichā* and divided by *Pramāṇa*, becomes the [required] answer, when the *Ichā* and the *Pramāṇa* are similar.

Bhāskara II [trans. Colebrooke 1817: 33, Sanskrit terms added], finally, states that

The first and last terms, which are the argument [*Pramāṇa*] and requisition [*Ichā*], must be of like denomination; the fruit [*Phala*], which is of a different species, stands between them.

Āryabhaṭa had given no corresponding explanation in terms of the similar and the non-similar (nor does the Bakhshālī manuscript, but it is anyhow outside the main Sanskrit stream in terminology, as we have seen). It thus seems as if the concepts have been adopted into the Sanskrit tradition around the onset of the seventh century. The very different ways in which the Sanskrit authors insert the observation shows that they do not copy one from the other.

That similarity is mentioned by Mahāvīra is a first argument that the concern with similarity originated in a vernacular environment (in economical transactions its relevance is obvious, in astronomical pure-number calculations it does not apply directly); that Bhāskara I introduces the observation in connection with a (commercial) example points in the same direction.⁷ Neither argument is more than a non-compulsory hint, however; no wonder that those who have worked exclusively on Indian material have never been taken aback by the seventh-century introduction of what might be nothing but a reasonable mathematical observation.

Late medieval Italy

Things look different, however, if we take the Italian “abbacus” school and its mathematics into account. The abbasus school was a school mainly for merchants’ and artisans’ sons, who frequented it for two years or less around the age of 12,

⁶ Thus at least according to the translation – always a slippery argument, even when a literal translation has been intended.

⁷ True, as a referee insists, the observation is likely to be quoted from a scholar and not directly from vernacular parlance; but since this scholar refers to it as something “the wise” *should* but apparently do not know, it appears either to be his own invention or a borrowing from non-scholars, and in any case to be related to a practice where the distinction similar/non-similar applies.

learning about calculation with Hindu-Arabic numerals and in general about basic commercial arithmetic – not least about the rule of three. The earliest references to the institution are from the 1260s, and the earliest textual witnesses of its mathematics from the outgoing thirteenth century.

One of the earliest formulations – perhaps the earliest one – presents the “rules of the three things”⁸ as follows in literal translation [Arrighi 1989: 9, trans. JH]:

If some computation was said to us in which three things are proposed, then we shall multiply the thing that we want to know with the one which is not of the same (kind), and divide in the other.⁹

Exactly the same formulation of the rule (except that multiplication is “against”, not “with”) is found in an anonymous *Liber habaci* [ed. Arrighi 1987b: 111] which can be dated to c. 1310.¹⁰ Already because this treatise uses no Hindu-Arabic but only Roman numerals (and fractions written with words), we may be sure that it is not derived from the *Livero*.

Two versions exist of Jacopo da Firenze’s *Tractatus algorismi*, originally written in 1307 but known from three fifteenth-century copies. In one of these (Vatican, Vat. lat 4826) the rule of three is slightly more elaborate [ed., trans. Høyrup 2007: 236f, error corrected]:

If some computation should be given to us in which three things were proposed then we should always multiply the thing that we want to know against the one which is not similar, and divide in the third thing, that is, in the other that remains.¹¹

The first example given runs as follows (*tornesi* are minted in Tours, *parigini* in Paris):

I want to give you the example to the said rule, and I want to say thus, vii *tornesi* are worth viiii *parigini*. Say me, how much will 20 *tornesi* be worth. Do thus, the thing that you want

⁸ *Le regole delle tre cose* – plural because separate rules are given according to the absence or presence of fractions.

The treatise in question is a *Livero de l’abbecho*, known from a fourteenth-century copy in the manuscript Florence, Ricc. 2404 [ed. Arrighi 1989]. Because of misinterpretation of copied internal evidence, the treatise has been wrongly dated to 1288–90. It is likely to be somewhat but not very much later [Høyrup 2005: 27–28, 47].

⁹ Se ce fosse dicta alchuna ragione ella quale se proponesse tre chose, sì devemo multiplicare quilla chosa che noie volemo sapere con quella che non è de quilla medessma, a partire nell’altra.

¹⁰ Gino Arrighi’s ascription to Paolo Gherardi (*fl.* 1328) is safely disregarded.

¹¹ Se ci fosse data alcuna ragione nela quale se proponesse tre cose, sì debiamo multiplicare sempre la cosa che noi vogliamo sapere contra a quella che non è simegliante, et parti nela terza cosa, cioè, nell’altra che remane.

to know is that which 20 *tornesi* will be worth. And the not similar is that which vii *tornesi* are worth, that is, they are worth 9 *parigini*. And therefore we should multiply 9 *parigini* times 20, they make 180 *parigini*, and divide in 7, which is the third thing. Divide 180, from which results 25 and $\frac{5}{7}$. And 25 *parigini* and $\frac{5}{7}$ will 20 *tornesi* be worth.¹²

The other two manuscripts (Milan, Trivulziana MS 90, and Florence, Riccardiana MS 2236), probably representing a revised version [ed. Høyrup 2007: 419, trans. JH], introduce the rule “If some computation should be *said* [*deta*] to us” (not “given”), while the rule itself turns around the final phrase, which becomes “... divide in the other, that is, in the third thing”;¹³ their formulation of the example only differs from that of the Vatican manuscript by using the phrase “the one which is not of the same”¹⁴ (that of the *Livero* etc.) instead of “the one which is not similar”.

Jacopo, an emigrated Florentine, wrote his treatise in Montpellier. Paolo Gherardi wrote his *Libro di ragioni* in the same place in 1328. *His* formulation is that of the *Livero* and of the *Liber habaci*. So is that of Giovanni de’ Danti’s *Tractato d’algorisimo* from 1370 [ed. Arrighi 1987a: 29], even though it copies much of its introduction (a high-flown general praise of knowledge) from Jacopo.¹⁵

The examples of all these treatises differ; their shared formulation of the rule is thus not the consequence of one author copying from the other; it must represent a formulaic expression which was in general circulation. It remained so for long – it is still found in the first printed commercial arithmetic (*Larte de labbaco*, also known as the “Treviso arithmetic” from 1478¹⁶), while Luca Pacioli presents us with a slight pedagogical expansion in the *Summa de arithmetica* [1494: fol. 57^r, trans. JH], already

¹² Vogliote dare l’exemplo ala dicta regola, et vo’ dire chosì, vij tornisi vagliono viiiij parigini. Dimmi quanto varranno 20 tornisi. Fa così, la cosa che tu voli sapere si è quello che varranno 20 tornisi. Et la non simegliante si è quello che vale vij tornisi, cioè, vagliono 9 parigini. Et però dobbiamo multiplicare 9 parigini via 20, fanno 180 parigini, et parti in 7, che è la terza chosa. Parti 180, che ne viene 25 et $\frac{5}{7}$. Et 25 parigini et $\frac{5}{7}$ varranno 20 tornesi.

¹³ ... partire nel’altra, cioè nella terza cossa.

¹⁴ quella che nonn’è di quella medesima.

¹⁵ It is of course possible that Jacopo copies from an unknown earlier source, which might then just be shared by de’ Danti (and the many others who have the same introduction). Given Jacopo’s early date this is not very likely (but anyhow unimportant for our argument).

¹⁶ Unpaginated, at least in my digital facsimile; but pp. 61f if the title page is page 1. In the end this treatise adds to the rule that the result will be of the nature of the non-similar thing, while the divisor will be similar.

present except for the words in {...} in his Perugia manuscript from 1478 [ed. Calzoni & Gavazzoni 1996: 19f]:

The rule of 3 says that the thing which one wants to know is multiplied by that which is not similar, and divided by the other {which is similar}, and that which results will be of the nature of that which is not similar, and the divisor will always be of the similitude of the thing which one wants to know.¹⁷

In both cases an alternative follows:

The same in other words. The rule of 3 says that the thing which is mentioned twice [*A* and *C* in the above letter formalism] should be looked for, of which the first is the divisor, and the second is multiplied by the thing mentioned once [*B*], and this multiplication is divided by the said divisor, and that which results from the said division will be of the nature of the thing mentioned once, and so much will the thing be worth {precisely} which we try to know.¹⁸

Jacopo and Pacioli were not the only ones to insert pedagogical expansions. Another example is found in Pietro Paolo Muscharelo's *Algorismus*, written in Nola¹⁹ in 1478 [ed. Chiarini et al 1972: 59, trans. JH]:

This is the rule of 3, which is the fundament for all commercial computations. And in order to find the divisor, always look for the similar thing, which is mentioned twice, and one of these will be the divisor, and I say that it will be the one which is not your request, and this your request you will get by multiplying with the other not similar thing, and this multiplication [i.e., product] you will have to divide by your divisor, and from it will come that which you will require.²⁰

¹⁷ La regola del 3 vol che se multiplichì la cosa che l'homo vol saper per quella che non e simigliante e partire per l'altra {che e simigliante} e quel che ne vene si ene de la natura de quella che non è simigliante {e sira la valuta de la cosa che volemo inquirere}. E sempre el partitor convien che sia de la similitudine de la chosa che l'homo vol sapere.

¹⁸ Idem sub aliis verbis. La regola del 3 vol che se guardi la cosa mentovata doi volte de le quali la prima è partitore, e la seconda se multiplica per la chosa mentoata una volta. E quella tal multiplicatione se parta per ditto partitore. E quello che ne vien de ditto partimento sira de la natura de la cosa mentovata una volta. E tanto varrà la chosa che cercamo sapere {aponto}.

¹⁹ In Campania, and thus outside the native ground of the abacus school, which may be the reason that it replaces the standard formula by an explanation.

²⁰ Questa si è la regola del 3, la quale è fondamento di omne ragione mercatantescha. Et per trovare il partitore guarda sempre la cosa simigliante la quale si è nominata dua fiате e una di quelle si serrà il partitore, e dico che serrà quella cosa la quale non serra tua dimanda e quella tua dimanda l'avverrai ad multiplicare coll'altra cosa non simigliante et quella multiplicatione averrai ad partire per lo tuo partitore e verràne quello che dimanderai.

As we see, Pacioli's reference to "the thing which is mentioned twice" is inserted here in the standard formula.

A last formulation to look at is found in Paolo dell'Abbaco's mid-fourteenth-century *Regoluzze* [ed. Arrighi 1966: 31, trans. JH], which does not formulate the rule as a merely arithmetical algorithm but prescribes a 2×2 organization on paper:

If you want to calculate, that is, to make computations of sale and purchase, write the thing [*materia*] in front of its price, and the similar below the similar; and then multiply these two numbers that are askew, and always divide by the number which is beside.²¹

As we see, not only the standard formula and its variations but also this practical prescription all circle around the concepts of the dissimilar and the similar.

Before we leave the Italian corpus, one weird aspect of the standard formula might be taken note of. The reference to *C* as "the thing that we want to know" is misleading: *C* itself is known, and that which we want to know is its counterpart (as made clear in the *Vedān'gajyotiśā*). But it would be the perfect translation of *icchā* or some corresponding vernacular Indian term; a loan translation is thus possible, though not very likely as long as no intermediate evidence connects the two.

Does any such intermediate evidence link the Indian and the Italian references to the similar/dissimilar?

Arabic sources

A first impression of the earliest extant Arabic description of the rule of three – the chapter on commercial transactions in al-Khwārizmī's *Algebra* from c. 820 CE – does not support any idea of transmitted formulations. It uses no name for the rule which might correspond to the Sanskrit or Italian reference to three things (actually, no name at all), and according to the best known modern translations it seems to build on the theory of proportions of *Elements* VII.

Frederic Rosen [1831: 68] translates as follows:

You know that all mercantile transactions²² of people, such as buying and selling, exchange and hire, comprehend always two notions and four numbers, which are stated by the enquirer; namely, measure and price, and quantity and sum. The number which ex-

²¹ Se vuoi chalculare, cioè fare ragione di vendita o di conpera; scrivi la materia di rinpetto al suo pregio, e lla simile sotto la simile; e poi multiplica queglii due numeri che stanno alla schisa, e parti per lo numero ch'è nel canto senpre.

²² The Arabic word is *mu'āmalāt*, referring to the economical transactions of social life in general, not only trade.

presses the measure is inversely proportionate to the number which expresses the sum, and the number of the price inversely proportionate to that of the quantity. Three of these four numbers are always known, one is unknown, and this is implied when the person inquiring says *how much?* and it is the object of the question. The computation in such instances is this, that you try the three given numbers; two of them must necessarily be inversely proportionate the one to the other. Then you multiply these two proportionate numbers by each other, and you divide the product by the third given number, the proportionate of which is unknown. The quotient of this division is the unknown number, which the inquirer asked for; and it is inversely proportionate to the divisor.

Roshdi Rashed agrees in his French translation [2007: 196] with Rosen that the translation must be made in terms of proportion theory but disagrees with Rosen in how to make the connection:

Sache que toutes les transactions entre les gens, de vente, d'achat, de change <de monnaies>, de salaire, et toutes les autres, ont lieu selon deux modes, et d'après quatre nombres prononcés par le demandeur, qui sont: quantité d'évaluation, taux, prix, quantité évaluée.

Le nombre qui est la quantité d'évaluation n'est pas proportionnel à celui qui est le prix. Le nombre qui est le taux n'est pas proportionnel au nombre de la quantité évaluée, et, parmi ces quatre nombres, trois sont toujours évidents et connus, et l'un d'eux est inconnu, qui, dans les termes de celui qui parle, est « combien », et qui est l'objet du demandeur.

On l'infère ainsi ; tu examines les trois nombres évidents ; il est nécessaire que, parmi eux, il y en ait deux, dont chacun n'est pas proportionnel à son associé. Tu multiplies les deux nombres évidents non proportionnels l'un par l'autre; tu divises le produit par l'autre nombre évident, dont <l'associé> non proportionnel est inconnu ; ce que tu obtiens est le nombre inconnu cherché par le demandeur, et qui n'est pas proportionnel au nombre par lequel tu as divisé.

Where Rosen finds “inversely proportionate”, Rashed thus sees “not proportional”. Neither makes much sense mathematically. The third modern translation, made by Boris Rozenfeld [1983: 45] therefore translates the critical term *mubāyin* neither as “not proportional” nor as “inversely proportional” but as *protiv*, “opposite”, probably thinking of a graphical 2×2 -scheme as described by Paolo dell'Abbaco. Both twelfth-century Latin translations, due respectively to Robert of Chester [ed. Hughes 1989: 64] and Gerard of Cremona [ed. Hughes 1986: 255], do the same, and Robert refers explicitly to the scheme. In their time, indeed, this scheme was well known, as made obvious by its use in the *Liber abbaci* (see below, n. 29).

Unfortunately, nothing in al-Khwārizmī's text suggests that he thought about such a scheme – but fortunately, a much more meaningful translation of *mubāyin* can be given [Wehr 1985: 131] – namely “different (in kind)” (or “dissimilar”), as also

indicated by Mohamed Souissi [1968: 96], with reference to precisely this passage. Al-Khwārizmī's terminology is thus related both to what turns up in India from Bhāskara I and Brahmagupta onward and to what we find in late medieval Italy.

Quite a few later Arabic authors do refer to the Euclidean theory – sometimes integrating it with the presentation of the rule of three, sometimes keeping the two topics separate though as neighbours. Al-Karajī's *Kāfī fi'l-ḥisāb* (c. 1010 CE) is an example of separate treatment [ed., trans. Hochheim 1878: II, 15–17, English JH]:

Chapter XLII. *Proportions*. Of the four magnitudes of the proportion, the first relates to the second as the third to the fourth. If you have found this correlation, then you obtain through interchange of the members that the first relates to the third as the second to the fourth. Further you also obtain, when combining, a proportion: the sum of the first and the second member relates to the second member as the sum of the third and the fourth member to the fourth. Further you may form differences [...].²³

If the first member is unknown, then you multiply the second by the third member and divide by the fourth. Similarly, if the fourth member is unknown, you divide this product by the first member. If the second or the third is unknown, then you multiply the first by the fourth and divide the product by the known one of the other two members.

If three numbers form a proportion [...]

Chapter XLIII. *Commercial transactions*. Know that in questions about commercial transactions you must have four magnitudes, which are pairwise similar, the price, the measure, the purchase amount and the quantity.²⁴

The price is the value of a measuring unit that is used in trade [...].

[...] Of these four magnitudes, three are always known, and one is unknown. You find the unknown magnitude by multiplying one of the known magnitudes, for instance the sum or the quantity, by that which is dissimilar to it, namely the measure or the price, and dividing the outcome by the magnitude which is of the same kind. What comes from it is the result.

Or if you prefer, put one of the known magnitudes, for instance the quantity or the sum [called the “purchase amount” a few lines earlier] in relation to the one that is similar to it [i.e., find their ratio], and thereby search the relation of the non-similar magnitude.²⁵

²³ The set of operations performed here presupposes that all four magnitudes are of the same kind. Al-Karajī thus has the good reasons of a good mathematician to keep proportions and rule of three apart.

²⁴ Corresponding, respectively, to Rosen's “price”, “measure”, “sum” and “quantity”.

²⁵ In our letter symbols, $Y = (^B/_A) X$. As we observe, this is not the rule of three but an alternative, here and in other Arabic works (e.g., ibn Thabāt, ed. [Rebstock 1993: 43]) called “by *niqba*”, “by relation”. It has the advantage of being intuitively easier to grasp, but the disadvantage of performing the division first, which will mostly increase rounding errors or entail difficult multiplications of fractional quantities.

It is clear already from the order of the four magnitudes involved (and corroborated by the whole formulation) that al-Karajī does not copy from al-Khwārizmī's exposition. This corresponds well to the presentation of algebra later in the treatise, which appears to draw on a pre-Khwārizmian form of that technique.²⁶

Ibn al-Bannā's concise *Talkhīṣ* (early 13th century CE) integrates the rule in the presentation of proportions [ed. trans. Souissi 1969: 87f, English JH]:

The four proportional numbers are such that the first is to the second as the third to the fourth.

The product of the first with the fourth is equal to the product of the second with the third.

When multiplying the first by the fourth and dividing the product by the second, one obtains the third. [...]

Whichever is unknown among these numbers can be obtained by this procedure from the other three, known, numbers. The method consists in multiplying the isolated given number, dissimilar from the two others, by that whose counterpart one ignores, and dividing by the third known number. The unknown results.

While al-Karajī and ibn al-Bannā³ are usually counted as “mathematicians”, ibn Thabāt was primarily a legal scholar, and his *Ghunyat al-ḥussāb* (“Treasures of the Calculators”, from around 1200 CE) is intended to teach the mathematics that could serve legal purposes. Even here, proportion theory and rule of three are integrated. The rule is stated thus [ed. trans. Rebstock 1993: 45, English JH]:

The fundament for all *mu'āmalāt*-computation is that you multiply a given magnitude by one which is not of the same kind, and divide the outcome by the one which is of the same kind.

This rule, as we see, is quite close to the one that was taught in the Italian abacus school. However, it precedes the earliest abacus treatises by at least half a century, and in any case it is difficult to imagine that a scholar teaching in the Baghdad *madrasah* should have direct access to what went on in Italy. We may safely assume that the rule he knew was widespread in a commercial community spanning at least the whole region from Iraq to ibn al-al-Bannā's Maghreb, and almost certainly also the traders of the Mediterranean as well as the Indian Ocean. Everywhere, it tended to penetrate even erudite presentations of the same subject-matter as a secondary explanation. *However, what penetrated the presentations of Brahmagupta, Mahāvīra*

²⁶ This follows in particular from his use of the key terms *al-jabr* and *al-muqābalah*, see [Høyrup 2007: 157], cf. [Saliba 1972].

and *Bhāskara I and II* can hardly have been the language of Arabic traders; it must have been the ways of Indian merchants and public officials speaking a Prakrit.

Latin presentations

This part of my story – the one that may convey information about Indian usages – turns out in the end to be, or at least to look, quite simple, and it would be tempting to use it as the final conclusion. However, this simplicity results from disregard of those features of the process that point away from it. Taking them into account will not refute the simple story, but they will show that there is more to the matter.

As we have seen, the two Latin translations of al-Khwārizmī’s *Algebra* made in Iberian area in the twelfth century both misunderstand his reference to the similar and dissimilar, but apart from that they are faithful to the original. Two other twelfth-century Latin works from the same area, the *Liber mahamalet* [ed. Vlasschaert 2010: II, 185, trans. JH] and the so-called “Toledan *Regule*” [ed. Burnett, Zhao & Lampe 2007: 155] have an approach which I know from nowhere else.²⁷ Of four numbers in proportion, the first and the fourth are declared “partners” (*socii*), and so are the second and the third. If one is unknown, then its partner shall divide any of the other two, and the outcome be multiplied by the third number – that is, the *nisba* approach or the seemingly similar rule $Y = (X/A) \times B$.²⁸ Afterwards, both specify differently (without observing that there is a difference), namely in agreement with the naked rule of three,

thus, if three are proposed and the fourth is unknown, multiply the second in the third, and divide what results by the first, and what comes out will be the fourth.

After a new headline “Chapter on buying and selling”, the *Liber mahamalet* [ed. Vlasschaert 2010: II, 186, trans. JH] repeats, but now recognizes that the methods are alternatives:

²⁷ The two texts are generally related, see [Burnett, Zhao & Lampe 2007: 145]. Nothing therefore forces us to believe that this shared peculiarity of theirs reflects a widespread pattern. (There is a slight similarity but nothing more with Abraham ibn Ezra’s *Sefer ha-mispar* [ed, trans. Silberberg 1895: 49].)

Even though the title *Liber mahamalet* clearly shows that the work is intended to present the *mu‘āmalāt* genre, this is not the only point where this treatise seems to explore matters on its own.

²⁸ This rule, corresponding to what was done in Mesopotamia, in Ancient Egypt and in Greek practical mathematics, will be in conflict with the Euclidean formulation in all practical applications, since A and X will not be “similar”. It was therefore avoided by those Arabic authors who wanted to base their calculations on Euclid.

When in buying or selling it is asked about something what is its price.

Do thus: Multiply the middle [number] by the last, and divide the product by the first.

Or divide the middle by the first, and what comes out of it multiply by the last, or divide the last by the first, and what comes out of it multiply by the middle. From all these modes results the unknown that is asked for.

The presentation of the matter in Fibonacci's *Liber abbaci* [ed. Boncompagni 1857: 83*f*, trans. JH] (from 1228, but at least this passage is likely to be close to the lost 1202 edition) looks like Fibonacci's personal way to describe what he has seen in use:

In all commercial exchanges [*negotiationes – mu'āmalāt?*], four proportional numbers are always found, of which three are known, but the remaining unknown. The first of these three known numbers is the number of sale of any merchandise, be it number, or weight, or measure [explanatory examples]. The second, however, is the price of this sale [...]. The third, then, will be the sale of some quantity of this merchandise, whose price, namely the fourth, unknown number, will not be known. Therefore, in order to find the unknown number from those that are known, we give a universal rule for all cases, namely, in the top of a board write the first number to the right, namely the merchandise.²⁹ Behind in the same line you posit the price of the same merchandise, namely the second number. The third too, if it is the merchandise, write it under the merchandise, that is, under the first, And if it is the price, write it under the price, that is, under the second. In this way, as it is of the kind of that under which it is written, thus it will also be of the quality or the quantity, whether in number, in weight or in measure. That is, if the superior number, under which one is writing is a number [of *rotul*³⁰], itself will also be *rotuli*, if pounds, pounds, [...]. When they are described thus, it will be obvious that two of those that are posited will always be contrary [*ex adverso*], which have to be multiplied together, and that if the outcome of their multiplication is divided by the third number, the fourth, unknown, will doubtlessly be found.³¹

²⁹ This prescription corresponds to inscription on an Arabic dust or clayboard (*takht* respectively *lawha*) – in agreement with the start from the right. Robert's and Gerard's translation of *mubāyin* as "opposite" shows that they think of the same scheme.

³⁰ A weight unit, see [Zupko 1981: 228].

³¹ In omnibus negotiationibus quattuor numeri proportionales semper reperiuntur, ex quibus tres sunt noti, reliquus uero est ignotus: primus quidem illorum trium notorum numerorum est numerus uenditionis cuiuslibet mercis, siue constet numero, siue pondere, siue mensura [...]. Secundum autem est pretium illius uenditionis [...]. Tertius uero quandoque erit aliqua eiusdem uendite mercis quantitas, cuius pretium, scilicet quartus numerus, ignoratur; et quandoque erit aliqua similis quantitas secundi pretii, cuius merces, scilicet quartus ignotus numerus, iterum ignorabitur. Quare, ut ignotus numerus per notos reperiatur, talem in omnibus tradimus regulam uniuersalem, uidelicet ut in capite tabule, in dextera parte scribas primum numerum, scilicet mercem; retro in eadem linea ponas pretium

As we see, Fibonacci knows the reason for speaking of the similar and the dissimilar, but it does not enter his prescription (which it would strain normal language to call a “rule”).

Much later in the work, namely within the long chapter 12 consisting of mixed problems [ed. Boncompagni 1857: 170, trans. JH], we find a problem which is solved by means of the rule of three but which *prima facie* seems to have nothing to do with a general presentation of that rule:

If it is asked about 6, to which number it has the same ratio [*proportio*] as 3 to 5, you do thus: Multiply 5 by 6, it will be 30; which divide by 3, 10 comes out of it, which is the number asked for; because as 3 is to 5, thus 6 is to 10. To be sure, we usually pose this question differently in our vernacular [*ex usu nostri vulgaris*]: namely that if 3 were 5,³² what then would 6 be? And just as it was said, 5 is similarly multiplied by 6, and the outcome divided by 3.³³

A similar problem follows, also given afterwards in “vernacular” terms. A total listing of the occurrences of the terms *vulgaris/vulgariter* in the work leaves no doubt that it refers to the usage of the precursor-environment for the abacus school, the community of commercial calculators working around the Mediterranean.

Iberia and Provence

This may seem strange: so far we have encountered nothing similar to this presumed “vernacular” way. But this is only because we did not look at Ibero-Provençal material

ipsius mercis, uidelicet secundum numerum; tertium quoque si fuerit mercis, scribe eum sub merce, scilicet sub primo; et si fuerit pretium, scribe eum sub pretio, uidelicet sub secundo; ita tamen, ut sicut fuit ex genere ipsius, sub quo scribendum est, ita etiam sit ex qualitate uel ex quantitate ipsius in numero, uel in pondere, uel in mensura; hoc est si superior numerus, sub quo scribendus est, fuerit numerus ipsorum, et ipse similiter fiat rotulorum; si librarum, librarum; [...]. Quibus ita descriptis, euidētissime apparebit, quod duo illorum positi erunt semper ex aduerso, que insimul multiplicentur, et summa multiplicationis eorum, si per reliquum tertium numerum diuidatur, quartus ignotus nimirum inuenietur.

³² “Were” (*essent*) is in the subjunctive. Fibonacci is fully aware that this question is counterfactual, 3 *is not* 5. All similar counterfactual questions below (in varying languages) are also stated in the subjunctive.

³³ Si queratur de 6, ad quem numerum eandem habeat proportionem, quam 3 ad 5, sic facies. Multiplica 5 per 6, erunt 30; que diuide per 3, exhibunt 10, que sint quesitus numerus; quia sicut 3 sunt ad 5, ita 6 sunt ad 10. Solent enim ex usu nostri uulgaris hanc eandem questionem aliter proponere: uidelicet ut si 3 essent 5; quid nam essent 6: et cum ita proponitur, multiplicantur similiter 5 per 6, et diuiditur summa per 3.

apart from what was written in Latin during the twelfth century, nor at what is probably the very earliest Italian abacus book.

Disregarding chronology, let us start in 1482 with Francesc Santcliment’s Catalan *Suma de la art de arismetica*. It introduces the *regla de tres* in these words [ed. Malet 1998: 163, trans. JH]:

It is called properly the rule of three, since within the said species 3 things are contained, of which two are similar and one is dissimilar. This said species is common to all sorts of merchandise. There is indeed no problem nor question, however tough it may be, which cannot be solved by it once it is well reduced.

And in our vernacular [*nostre vulgar*] the said species begins: If so much is worth so much, what will so much be worth?

The solution of this rule is commonly said: Multiply by its contrary and divide by its similar.³⁴

First, of course, we observe the reference to the “vernacular” connected to almost the same phrase, though no longer “counterfactual” (something “being” something which it is not); instead, one thing is now “worth” another thing abstractly, that is, in pure numbers. There are also references to the “similar” and the “dissimilar”, but the “common” formulation of the solution “Multiply by its contrary and divide by its similar” does not coincide precisely with the Italian standard abacus rule. In spite of the shared reference to the vernacular, everything remains so different from the text of the *Liber abbaci* that any copying or direct inspiration from that work can be excluded.

A thorough inspection of all known commercial arithmetics of abacus type written in Ibero-Provençal area until 1500 will show that they share the counterfactual or abstract “being-worth” formulation of the rule (from now on presented in chronological order).

The earliest of these treatises is a Castilian *Libro de arismética que es dicho alquarismo*, known from an early-sixteenth-century copy of an original written in 1393. Some aspects call to mind the *Liber mahamaleth*, showing the *Libro ... dicho alquarismo* to be partially rooted in an Iberian tradition going back to the Arabic

³⁴ [S]egueix se la sisena specia: que s nomena regla de tres. Diu se propriament pertant regla de tres: per quant dins la dita specia se contenen 3 coses. de les quals les dues son semblants e la una es dessemblant. La qual specia es general en tota mercaderia. Car no es nenguna rao ne questio per fort que sia: que per aquesta specia essent be reduida no sia absoluta.

E comença la dita specia en nostre vulgar si tant val tant: que valra tant.

La absolucio de aquesta regla que comunament se diu multiplica per son contrari e parteix per son semblant.

period – especially use of “ascending continued fractions” ($\frac{a}{n}$ and $\frac{b}{p}$ of $\frac{1}{n}$ and ...). Most aspects, however, and in particular the presentation of the rule of three [ed. Caunedo del Potro & Córdoba de la Llave 2000: 147, trans. JH] are quite different. This presentation combines the counterfactual with the abstract “being worth”, and has no hint of a graphical organization in a 2×2 -scheme (instead, the same linear organization is used as in the Bakhshālī manuscript, but this is too close at hand to be taken as evidence of any link):

This is the 6th species, which begins “if so much is worth so much, what will so much be worth”.

Know that according to what the art of algorism commands, to make any calculation which begins in this way, “if so much was so much, what would so much be?”, the art of algorism commands that you multiply the second by the third and divide by the first, and that which comes out of the division, that is what you ask for. As if somebody said, “if 3 were 4, what would 5 be?”, in order to do it, posit the figures of the letters³⁵ as I say here, the 3 first and the 4 second and the 5 third, 3, 4, 5, and now multiply the 4, which is the second letter, with the 5, which is the third, and say, 4 times 5 are 20, and divide this 20 by the 3, which stands first, and from the division comes $6\frac{2}{3}$, so that if they ask you, “if 3 were 4, what would 5 be?”, you will say $6\frac{2}{3}$, and by this rule all calculations of the world are made which are asked in this way, whatever they be.³⁶

Next in time comes the “Pamiers Algorithm” from c. 1430 [Sesiano 1984: 27]. Jacques Sesiano offers a partial edition only, for which reason I cannot quote the whole introduction – but he does show [1984: 45] that it follows the pattern “ $4\frac{1}{2}$ is worth $7\frac{2}{3}$, what is $13\frac{3}{4}$ worth?”.

The anonymous mid-fifteenth-century Franco-Provençal *Traicté de la pratique d'algorisme* also follows the same general pattern but is never so close to the others that direct copying can be suspected. Its presentation of the rule of three [ed. Lamassé 2007: 469, trans. JH] runs thus:

³⁵ On p. 134 the author explains “the letters of algorism” to be the Hindu-Arabic numerals.

³⁶ Esta el la 6 especie que comienza sy tantas valen tantas qué valdrán tantas.

Sabe que segund manda la regla del algarismo, para fazer qualquier cuenta que se pregunte por esta manera, en que pregunten asy, sy tanto fase tanto ¿qué sería tanto? Manda la regla del arte del algarismo que multipliques lo 2º por lo 3º e partirás por lo primero e a lo que saliere a la parte, aquello es lo que preguntan, asy como quien dixiese, sy 3 fuesen 4 ¿qué sería 5?, para lo faser, pon las figuras de las letras como aquí dize, el 3 primero y el 4 y el 2º y el 5 el tercero, 3, 4, 5, y agora multiplica el 4 que es la segunda letra con el 5 que es la tercera e di 4 veces 5 son 20 e estos 20 pártelos por el 3 que están primero e verná a la parte $6\frac{2}{3}$, asy que sy te preguntaren que sy 3 fuesen 4 ¿qué sería 5?, dirás que $6\frac{2}{3}$, e por esta regla se farán todas las cuentas del mundo qualesquier que sean que se pregunte por esta manera.

This rule is called rule of three for the reason that in the problems that are made by this rule three numbers are always required, of which the first and the third should always be similar by counting one thing. And from these three numbers result another one, which is the problem and conclusion of that which one wants to know. And it is always similar to the second number of the three. By some this rule is called the golden rule and by others the rule of proportions. The problems and questions of this rule are formed in this way: “If so much is worth so much, how much will so much be worth?”. As for example, “if 6 are worth 18, what would 9 be worth?”. For the making of such problems there is such a rule:

Multiply that which you want to know by its contrary and then divide by its similar. Or multiply the third number by the second and then divide by the first.³⁷

As we see, this version emphasizes the similar and the dissimilar, and combines the linear arrangement of the Castilian *Libro de arismética* with the formulation we know from Santcliment.

Closely connected to this *Traicté* is Barthélemy de Romans’ *Compendy de la pratique des nombres*.³⁸ It says about the rule of three [ed. Spiesser 2003: 255–257, trans. JH] that it is “the most profitable of all”, and gives two rules, one for finding Y from A , X and B , and one probably meant for finding B from A , X and Y ,

Multiply that which you want to know by its contrary, and then divide by its similar,³⁹

and

Multiply that which you know by that which is wholly dissimilar to it, and then divide by its similar,⁴⁰

after which it goes on with the composite rules. The first of these rules, we see, is shared with the *Traicté* and with Santcliment; the second, by using the term dissimilar

³⁷ Ceste regle est appellee regle de troys pource que es raisons qui se font par ceste regle sont tousiours requis troys nombres desquelz le premier et le tiers doivent tousiours estre semblants en nombrant une chose. Et diceulz troys nombres en resulte ung autre qui est la raison et conclusion de ce que l’on veult savoir. Et est tousiours semblable au second nombre des troys. Ceste regle selon aucuns est appellee regle doree et selon autres regle des proportions. Les raisons et questions de ceste regle se forment en ceste maniere. Si tant vault tant que vaudra tant ? Comme par exemple : se 6 valent 18 que vaudront 9 ? Pour faire telles raisons il en est une telle regle.

Multiplie ce que veulz savoir par son contraire et puis partiz par son semblant. Ou multiplie le tiers nombre par le second et puis partiz par le premier.

³⁸ Probably written around 1467 but only known from a revision made by Mathieu Préhoude in 1476.

³⁹ Multiplie ce que veulz savoir par son contraire et puis partiz par son semblant.

⁴⁰ Multiplie ce que sceiz par la chose qui luy est de tout dissemblant et puis partiz par son semblant.

(*dissemblant*) instead of contrary, looks as if it was of Italian inspiration (it might thus simply be an alternative formulation of the rule for finding Y from A , X and B). The first example, however, is in purely Iberian tradition, “if 5 is worth 7, what is 13 worth?”.

The final Ibero-Provençal treatise is Francés Pellos’s *Compendion de l’abaco*, printed in Nice in 1492. It starts by a general introduction to the theme [ed. Lafont & Tournerie 1967: 101–103, trans. JH], that does not look in detail like anything else we have seen except in the concluding *General rule to find every thing*, and which in its entirety is likely to be Pellos’s own description of the situation:

In this chapter I want to give you a good mode and way in which you can always quickly and without great toil find all things that you want to buy or sell. And know that this chapter is called the chapter and rule of three things. In every computation of trade three numbers are indeed necessary.

The first number.

The first number is always the thing bought or sold, and you need to keep it well in memory.

The second number.

Know that the second number shall always be the value or the price of that which you have bought or sold.

The third example or number.

And the third number shall always be the thing that you want to know, that is to say, the thing that you want to buy.

Remember that the first and the third numbers are always the same thing.

And know further that the first number and the third shall always be one thing. And if they are not certainly one thing, then you shall reduce them to a form where they speak of one thing, or matter, for in no way on earth they must not be different, as appears afterwards in the examples.

General rule to find every thing.

Always multiply the thing that you want to know by its contrary. And the outcome of this multiplication you divide by its similar, and that which comes out of such a division will be the value of the thing that you want to know.

[section on reduction of units]

*This is the way how you should say in matters that ask: if so much is worth so much, how much is so much worth? In this way, you may understand more clearly in the following examples.*⁴¹

⁴¹ In aquest capitol yeu ti voli donar brava manera he via per laqual tu l’augiament en general podes prestament trobar sensa grand fatica totas causas che voles comprar aut revendre. Et sapias che aquest capitol s’apella lo capitol et regula de tres causas. Car en cascun rason de merchantias son necessaris tres nombres.

Lo prumier nombre.

The first examples that follow ask “if 4 are worth 9, what are 5 worth?”, “if 3 and a half is worth 6, how much are 4 worth?”, etc. After six similar examples follows a graphical scheme, deceptively similar to the one we know from Paolo dell’Abbacho and Fibonacci but actually used for reducing rule-of-three-type problems involving fractions into problems involving only integers (and too similar to many other schemes used in abacus manuscripts to be supposed with any degree of certainty to be inspired by the traditional rule-of-three diagram).

Italy revisited

As we see, all Iberian and genuinely Provençal presentations of the rule use the counterfactual or abstract being-worth formulation; from the mid-fifteenth century *Traicté* onward they also know the notions of similar/dissimilar – whether because of interaction with the Italian tradition or because of other inspirations cannot be decided.

Many Italian abacus treatises, on the other hand, also know the counterfactual problem – and even “counterfactual *calculations*”, such as “If 9 is the 1/2 of 16, I ask you what part 12 will be of 25”, found in Gherardi’s *Libro di ragioni* [ed. Arrighi 1987b: 17, trans. JH]. But as was the case in the *Liber abbaci* (which by the way also contains counterfactual calculations), counterfactual problems and calculations are

Lo prumier nombre tostemps es causa comprada aut venduda, et es necessari che tengas ello ben en memoria.

Lo segont nombre.

Sapias che lo segont nombre deu tostemps esser lo valor aut lo pres de sso che aves comprat aut vendut.

Lo ters exemple ho nombre.

Et lo ters nombre tostemps deu esser la causa che demandas assaber so es a dire la causa che tu voles comprar.

Memoria per esser lo prumier et lo ters nombres tostemps una causa.

Item mayas sapias che lo prumier nombre et lo ters tostemps devon esser una causa. Et si non eran certanament una causa, adunques debes ellos redure en forma che parlon de una causa, aut materia, car en nenguna manera del monde non devon esser differens, coma apres appar en los exemples.

Regula general a trobar tota causa.

Multiplica tostemps la causa che demandas assaber per son contrari. Et la summa d’aquella multiplicacion partas per son semblant, et so che per tal partir ven, sera lo valor d’aquella causa che demandas assaber.

[...]

Aquesta es la mainera coma tu debes dire en causas que demandas : si tant val tant, quant val tant? Per aquesta mainera, tu podes entendre plus clar per aquest exemple sequent.

always found long after the presentation of the rule of three, or as illustrations of the rule following after many examples of the ordinary commercial kind.⁴² They are clearly meant to be recreational, and do not belong to the basic didactical stock (for which reason they are invariably counterfactual, not of the abstract being-worth type).

There are *two* exceptions to this rule. The first of these is the apparently earliest Italian abacus treatise, the “Columbia Algorithm” [ed. Vogel 1977], the original of which is likely to have been written around 1285–90 (what we have is a fourteenth-century copy).⁴³ The rule of three is approached in two different ways.

On one hand, there is a general presentation of the rule, not mentioning the name “rule of three” [ed. Vogel 1977: 39f, trans. JH]:

Remember, that you cannot state any computation where you do not mention three things; and it is fitting that one of these things must be mentioned by name two times; remember also that the first of the things that is mentioned two times by name must be the divisor, and the other two things must be multiplied together.

This is followed by an example dealing with the exchange of money. Later this formulation is used a couple of times [ed. Vogel 1977: 48, 50] in examples which explicitly speaks of the “rule of the three things”. We recognize Pacioli’s second formulation of the rule, which must thus have survived somewhere in the intervening two hundred years, even though I have not noticed it in texts I have looked at.

Mostly when the rule is used, however, a problem is reduced to a counterfactual [ed. Vogel 1977: 31f, 57, 61, 64f, 70 83, 80, 83, 86, 90, 109, 111, 123f] or an abstract being-worth [ed. Vogel 1977: 52, 112] formulation (“It is as if you said, ‘if a were/is worth b , what ...’”). At times [ed. Vogel 1977: 52, 57, 64, 83], the rule is also mentioned by name in these connections.

Finally, a number of times the rule is called by name but without any reference to either the “mentioned”, the counterfactual or the abstract being-worth formulations [ed. Vogel 1977: 52, 54f, 58, 110].⁴⁴ All in all, it is fairly obvious that the compiler of the treatise knows not only the Iberian “vernacular” way but also the idea underlying

⁴² This is where Jacopo [ed. trans. Høyrup 2007: 238] asks the question “if 5 times 5 would make 26, say me how much would 7 times 7 make at this same rate”.

⁴³ The dating of the treatise is discussed in [Høyrup 2007: 31 n. 20].

⁴⁴ The first folios of the treatise are missing, and so is the folio preceding the general introduction to the rule in “mentioned”-formulation. It is therefore impossible to exclude that a general introduction to the rule in one of the Ibero-Provençal formulations was present in the original. Nor must this necessarily have been the case, however.

what was to become the Italian standard formula, expressing it however differently (the two “similar” things becoming that which is mentioned “two times”).

The treatise stands at the very beginning of the Italian abacus tradition, and it is thus not strange that it draws on discordant sources. In other respects too it has links to the Iberian tradition as we know it from the *Libro de arismética que es dicho alquarismo* – cf. for instance [Høyrup 2007: 85]. At the same time it makes use the Maghreb notation for ascending continued fractions (the *Libro ... dicho alquarismo* knows the fraction type but not the Maghreb notation).⁴⁵ However, there are few traces in later times of its idiosyncrasies it was copied at some moment during the fourteenth century; in 1344, Dardi of Pisa shares its (mis)use of the notation for ascending continued fractions, writing $\frac{2}{c}\frac{1}{2}$ for “2 *censi* and $\frac{1}{2}$ of a *censo*” [Høyrup 2010b: 23]; as we have seen, finally, the “mentioned” formulation of the rule of three turns up in two manuscripts from 1478 (Pacioli and Muscharello); but that is all I have observed.

Interestingly, Muscharello’s treatise (written, we remember, outside the homeland of the abacus tradition) is the other exception to the rule that the Italian treatises never use counterfactual problems to introduce the rule of three. Indeed, its first example [Chiarini et al 1972: 9] asks “if two were 3, what would five be?”⁴⁶ However, the exception is minimal – immediately afterwards, the treatise explains the meaning by an example of the usual kind, “it is as saying, if two palms of cloth are worth 3 *tari*, what will 5 palms be worth”, and solves this problem, not the one it explains. It seems a good guess that the counterfactual introduction was inherited together with the “mentioned”-structure, if not from the Columbia algorithm then from a shared source – but then, as we see, almost discarded, in harmony with the integration of the “mentioned”-structure in the standard-formulation of the rule.

Apart from the Columbia algorism, no later Italian treatise I know of thus used the counterfactual or the abstract being-worth structures as the basic representation of or model for the rule of three.

⁴⁵ This notation had also been used by Fibonacci, but Fibonacci is not a likely source. Firstly, Fibonacci always writes these fractions from right to left, as do the Maghreb writers; the writing direction in the Columbia algorism alternates. Secondly, the Columbia algorism sometimes uses the notation when q is not a denominator but a metrological unit, a thing Fibonacci would never do (he knows that a denominator is a divisor, not a denomination). Thirdly, the Columbia algorism has nothing else in common with the *Liber abbaci* (not to speak of Fibonacci’s more sophisticated works).

⁴⁶ Se doi fosse 3, che sserria cinque?

What is the origin of the Ibero-Provençal representation?

As we have seen, Fibonacci refers to the counterfactual structure as the “vernacular” representation of the rule of three. Around 1200 it must hence have been widespread at least in some part of the Mediterranean environment he knew. Where?

We have no solids hints. For linguistic reasons, the counterfactual structure can hardly be Arabic: since the copula is not expressed in Semitic languages, “*a* is *b*, what is *c*?” should correspond to the opaque “*a b, c* what?”. The abstract being-worth formulation, on the other hand, is obviously possible in Arabic, and we do have a couple of Arabic texts which come near to it. In the probably most faithful version of al-Khwārizmī’s algebra – Gherardo’s translation [ed. Hughes 1986: 256] – the first example deals with a commercial problem, “10 *qafīz* are for six dragmas, what do you get for four dragmas?”;⁴⁷ the second, however, is abstract “`ten are for eight, how much is the price for four?’, or perhaps it is said, `four of them are for which price”.⁴⁸ The words “perhaps it is said” suggest that al-Khwārizmī quotes a common way of speaking – but we cannot be sure that this refers precisely to the abstract aspect of the formulation.

However, Robert’s translation [ed. Hughes 1989: 65] as well as the extant Arabic version [ed. Rashed 2007: 198] present the first example already in abstract formulation. Both represent the text as it developed in use, and the change could thus reflect common parlance. On the other hand, the step from concrete to abstract formulation is easily made, and the very scattered occurrences of similar wordings in Arabic texts (and their absence from the *Liber mahamalet* and the “Toledan *regule*”) do not suggest that they represent a widespread vernacular.⁴⁹ The safest assumption is that the

⁴⁷ Decem cafficii sunt pro sex dragmis; quot ergo provenient tibi pro 20 quattuor dragmis?

⁴⁸ ‘Decem sunt pro octo; quantum est pretium quattuor?’ Aut forsitan dicitur; ‘Quattuor eorum quanti pretii sunt’.

⁴⁹ Ibn al Khiḍr al Quraṣī (Damascus, mid-eleventh century) explains [ed., trans. Rebstock 2001: 64, English JH] the foundation for “sale and purchase” to be the seventh book of Euclid, after which he states (emphasis added) that “this corresponds to *your* formulation, `So much, which is known, for so much, which is known; how much is the price for so much, which is also known?’”. The reference to “your formulation” suggests that a general way of speaking about the matter is referred to. But was this general way already abstract, or has the abstraction been superimposed by al-Quraṣī?

In [Høyrup 2010a: 13], I took an isolated quotation from A. S. Saidan’s translation of al-Baghdādī as a hint of Persian counterfactual usage. Since then Mahdi Abdeljaouad has got access to the book and translated the whole surrounding passage (for which my sincere thanks!). In context,

abstract being-worth shape and its counterfactual variant were widespread at least somewhere in the Iberian world toward 1200;⁵⁰ before that we do not know.

So, whereas the first part of my story had few loose ends, these abound in the second part. Nobody should be surprised: loose ends are always there, and we can only create a simple coherent story by disregarding them – which does not necessarily make the coherent story untrue, only incomplete.

(And then I have not even touched at the
presence of the rule of three in
China, another intriguing
but unapproachable
loose end.)

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the use of the Persian expressions *dah yazidah*, “ten (is) eleven”, and *dah diyazidah*, “ten (is) twelve” in calculations of profit and loss turn out to have no such implications. Cognitively, what goes on is rather an analogue of the medieval Italian (and modern) notion of *per cento*, “percent”.

⁵⁰ It should be remembered that the 13th-century Vatican manuscript Palat. 1343 of the *Liber abaci* refers (fol. 47^{r-v}, new foliation) to use of a work by a “Castilian master”. Boncompagni knew [1851: 32], but since this early manuscript is incomplete, he used a later one (from where this observation is absent) for his edition.

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